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Independent domination in triangle-free graphs

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Abstract

Let G be a simple graph of order n and minimum degree δ. The *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of vertices of G. We establish upper bounds, as functions of n and $\delta \leqslant n/2$, for the independent domination number of triangle-free graphs, and over part of the range achieve best possible results.

Keywords: Independent domination; Triangle-free graphs

1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$ and minimum degree δ. An *independent set* is a set of pairwise non-adjacent vertices of G. A subset I of V is a *dominating set* if every vertex of $V - I$ has at least one neighbour in I. The *independent domination number* $i(G)$ is defined to be the minimum cardinality among all maximal independent sets of G. An independent set is maximal if and only if it is dominating, so $i(G)$ is also the minimum cardinality of an independent dominating set in G. This graph-theoretical invariant has been much studied in the literature, see for example [5].

A number of previous papers on the parameter $i(G)$ have focussed upon finding upper bounds as functions of n and δ, including Favaron [1], the present author [2] and Sun and Wang [7]. In view of their relevance to the current study, we summarise these results here.

Proposition 1 (*Favaron [1], Haviland [2], Sun and Wang [7]*). *Any simple graph G of order n and minimum degree δ satisfies*

$$i(G) \leqslant \begin{cases} n + 2\delta - 2\sqrt{n\delta} & \text{if } 0 \leqslant \delta \leqslant n/4, \\ 2(n-\delta)/3 & \text{if } n/4 \leqslant \delta \leqslant 2n/5, \\ \delta & \text{if } 2n/5 \leqslant \delta \leqslant n/2, \\ n - \delta & \text{if } n/2 \leqslant \delta \leqslant n-1. \end{cases}$$

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Note that Proposition 1 implies $i(G)\leqslant n/2$ for $\delta\geqslant n/4$. Moreover all of the upper bounds, with the exception of those in the range $n/4<\delta<2n/5$, are best possible. The bound for $0\leqslant\delta\leqslant n/4$ was conjectured by Favaron [1] and proved by Sun and Wang [7]. Over this range of δ, the following graphs are extremal: for δ and $\ell\geqslant 2$ positive integers, let $F(\delta,\ell)$ be the family of graphs such that $V=\bigcup_{j=1}^{\ell}(S_j\cup F_j)$, where $|F_j|=\delta$, $|S_j|=\delta(\ell-1)$ and $xy\in E$ if and only if $x\in S_j$, $y\in F_j$ or $x\in F_j$, $y\in F_k$, $j\neq k$. The results for $n/4\leqslant\delta\leqslant n/2$ were given by the present author [2]. Favaron [1] established the upper bound for $\delta\geqslant n/2$ and showed that it is attained only by complete multipartite graphs with vertex classes all of the same order. The analogous problem for regular graphs was considered by the present author [3,4] and Lam et al. [6].

Motivated by these earlier investigations, the aim of this paper is to provide upper bounds for the independent domination number of triangle-free graphs, as functions of n and δ. Clearly no pair of adjacent vertices in such graphs can have a common neighbour, and it is this property which we exploit in many of our arguments.

In restricting our study to triangle-free graphs, it is easily seen that $\delta\leqslant n/2$, for otherwise $|E|>n^2/4=|e(K_{n/2,n/2})|$ and so $G\supset K_3$ by Turán's Theorem. Furthermore, as observed in [2], for $2n/5\leqslant\delta\leqslant n/2$ the complete bipartite (and hence triangle-free) graphs $K_{\delta,n-\delta}$ have independent domination number δ and so, by Proposition 1, are extremal amongst graphs in general. Consequently, it suffices to establish upper bounds for the cases $0\leqslant\delta\leqslant n/4$ (Theorem 4) and $n/4<\delta<2n/5$ (Theorem 7). Where our results are best possible, we cite examples of corresponding extremal graphs. Where there is scope for improving our upper bounds, we present a lower bound (Theorem 8) and an associated conjecture.

In what follows, we abbreviate $i(G)$ to i where it is unambiguous. The *open neighbourhood* in G of a vertex $v\in V$ will be denoted by $N(v)=\{u\in V : uv\in E\}$, and that of a set of vertices $X\subset V$ by $N(X)=\bigcup_{x\in X}N(x)\cap(V-X)$.

2. Results

By Proposition 1, it is possible that $i>n/2$ only if $\delta<n/4$. The proof of our main theorem for these smaller values of δ requires two preliminary lemmas. In all ensuing work, unless indicated otherwise, let G be an extremal graph containing a minimum maximal independent set I.

Lemma 2. *Let G be a simple, triangle-free graph of order n and minimum degree $\delta<n/4$. If $i>n/2$ then any pair of adjacent vertices in $V-I$ have disjoint neighbourhoods in I with union of order at most $n-i-\delta$.*

Proof. Suppose $i>n/2$ and that there exist $x_1,x_2\in V-I$ with $x_1x_2\in E$. For $j=1,2$, let $W_j=N(x_j)\cap I$. Writing Δ for the maximum degree of G, Proposition 2 of [2] shows that $i\leqslant n-\Delta$. Hence $|W_j|\leqslant\Delta\leqslant n-i<i$, so no vertex of $V-I$ is joined to all of I.

Form the set $X_j=\{v\in V-I : N(v)\cap I\subseteq W_j\}$ and let R_j be a maximal independent set of $G[X_j]$ containing x_j. As G is triangle-free, observe that x_j is isolated in $G[X_j]$ with $W_1\cap W_2=\emptyset$ and $X_1\cap X_2=\emptyset$. The set $R_j\cup(I-W_j)$ is maximal independent for G, so $|R_j|+(i-|W_j|)\geqslant i$, implying

$$|X_j|\geqslant|R_j|\geqslant|W_j|. \tag{1}$$

If $|W_1\cup W_2|=|W_1|+|W_2|>n-i$ then by (1) we have $|V-I|\geqslant|X_1|+|X_2|\geqslant|W_1|+|W_2|>n-i$, an obvious contradiction. Thus $|W_1\cup W_2|\leqslant n-i$, so $|I-(W_1\cup W_2)|\geqslant i-(n-i)=2i-n>0$ and $I-(W_1\cup W_2)\neq\emptyset$. Now $N(I-(W_1\cup W_2))\subseteq V-I-(X_1\cup X_2)$, because if any vertex of $I-(W_1\cup W_2)$ has a neighbour in X_j, then this contradicts the fact that $N(v)\cap I\subseteq W_j$ for all $v\in X_j$. Since each vertex of $I-(W_1\cup W_2)$ has at least δ neighbours in $V-I$, applying this fact in conjunction with (1) gives

$$n-i=|X_1|+|X_2|+|V-I-(X_1\cup X_2)|\geqslant|W_1|+|W_2|+\delta,$$

which proves the lemma. □

Lemma 3. *Let G be a simple, triangle-free graph of order n and minimum degree $\delta<n/4$. If $i>n/2$ then any pair of non-adjacent vertices in $V-I$ have neighbourhoods in I with union of order at most $n-i-\delta$.*

Proof. Suppose $i > n/2$ and consider any non-adjacent pair of vertices $x_1, x_2 \in V - I$. For $j = 1, 2$, construct the sets W_j and X_j in a manner similar to the proof of Lemma 2. Note that in this case x_j is isolated in $G[X_j]$, but we may not assume that $W_1 \cap W_2 = \emptyset$ and $X_1 \cap X_2 = \emptyset$.

If $|W_1 \cup W_2| > n - i$, then we can form a maximal independent set for G, including x_1 and x_2, of order at most $n - |W_1 \cup W_2| < n - (n - i) = i$, which contradicts the minimality of I. Hence $|W_1 \cup W_2| \leqslant n - i$ and, as in the proof of Lemma 2, we know $|I - (W_1 \cup W_2)| \geqslant i - (n - i) = 2i - n > 0$, so $I - (W_1 \cup W_2) \neq \emptyset$. Again $N(I - (W_1 \cup W_2)) \subseteq V - I - (X_1 \cup X_2)$, with each vertex of $I - (W_1 \cup W_2)$ having at least δ neighbours.

This time we can form a maximal independent set I^* of G, including x_1, x_2 and some $y \in I - (W_1 \cup W_2)$, of order

$$|I^*| \leqslant n - |W_1 \cup W_2| - |N(y)| \leqslant n - |W_1 \cup W_2| - \delta.$$

Since $|I^*| \geqslant i$ then $|W_1 \cup W_2| \leqslant n - i - \delta$, as claimed. $\square$

Theorem 4. *Any simple, triangle-free graph G of order n and minimum degree δ satisfies*

$$i \leqslant \begin{cases} n + 2\delta - 2\sqrt{n\delta} & \text{if } 0 \leqslant \delta \leqslant 16n/121, \\ n + 3\delta - 2\sqrt{\delta(n+3\delta)} & \text{if } 16n/121 \leqslant \delta \leqslant n/6, \\ n/2 & \text{if } n/6 \leqslant \delta \leqslant n/4. \end{cases}$$

Proof. Suppose $0 \leqslant \delta \leqslant n/4$. If $\delta = n/4$ then Proposition 1 implies $i \leqslant n + 2\delta - 2\sqrt{n\delta} = n/2$, as required.

Prior to giving the main argument of the proof, we state the following three facts, each of which is easily verified:

(F1) $n + \delta - 2\sqrt{n\delta} \geqslant n/2$ if and only if $\delta \leqslant (3 - 2\sqrt{2})n/2$.
(F2) $n + 3\delta - 2\sqrt{\delta(n+3\delta)} \geqslant n/2$ if and only if $\delta \leqslant n/6$.
(F3) $n + \delta - 2\sqrt{n\delta} \leqslant n + 3\delta - 2\sqrt{\delta(n+3\delta)}$ if and only if $\delta \geqslant 0$.

Note that the minimum value of the three upper bounds given in the statement of the theorem, over the ranges of δ specified, is $n/2$. Thus if $n + \delta - 2\sqrt{n\delta} < i \leqslant n/2$ then the conclusion of the theorem holds and there is nothing to prove. Similarly, if $i \leqslant n + \delta - 2\sqrt{n\delta}$ and $n + \delta - 2\sqrt{n\delta} \geqslant n/2$ then by (F1), $\delta \leqslant (3 - 2\sqrt{2})n/2$ and again the result holds. Therefore, we may assume that $i \geqslant \max\{n/2, n + \delta - 2\sqrt{n\delta}\}$ and $\delta < n/4$.

Choose $x \in V - I$ such that $w = |N(x) \cap I|$ is maximal, and form the sets W, X and R as previously. Again we deduce that $R \cup (I - W)$ is maximal independent for G, so $|X| \geqslant |R| \geqslant w$ and

$$n - i = |X| + |V - I - X| \geqslant w + |V - I - X|. \tag{2}$$

First, consider the case $w \leqslant n - i - \delta - w$. Now $N(I - W) \subseteq V - I - X$, with each vertex of $I - W$ having at least δ neighbours. In addition, each vertex of $V - I - X$ has at most w neighbours in $I - W$, so substituting in (2) we obtain

$$n - i \geqslant w + (i - w)\delta/w.$$

As a function of w, the right-hand side of this inequality minimises at $w = \sqrt{i\delta}$. However, $\sqrt{i\delta} \geqslant (n - i - \delta)/2$ for $i \geqslant n + \delta - 2\sqrt{n\delta}$, which holds by assumption. (In order to verify the last claim, square both sides of the first inequality; then rearrange and solve the resultant quadratic inequality for i to yield the lower bound stated.) Thus in fact the right-hand side minimises at $w = (n - i - \delta)/2$, giving $i \leqslant n + 3\delta - 2\sqrt{\delta(n+3\delta)}$.

If not, we must have $w > n - i - \delta - w$. By Lemmas 2 and 3, all vertices of $V - I - X$ have at most $n - i - \delta - w$ neighbours in $I - W$. Also, each vertex of $I - W$ has at least δ neighbours in $V - I - X$, so substituting in (2) gives

$$n - i \geqslant w + (i - w)\delta/(n - i - \delta - w).$$

The right-hand side of this expression is an increasing function of w, and so minimises at $w = (n - i - \delta)/2$, again yielding $i \leqslant n + 3\delta - 2\sqrt{\delta(n+3\delta)}$.

Given our initial assumption and inequalities (F1)–(F3), we have thus shown that $i \leqslant n + 3\delta - 2\sqrt{\delta(n+3\delta)}$ for $0 \leqslant \delta \leqslant n/6$ and $i \leqslant n/2$ for $n/6 \leqslant \delta \leqslant n/4$. Finally, Proposition 1 states that $i \leqslant n + 2\delta - 2\sqrt{n\delta}$ for general graphs over the range $0 \leqslant \delta \leqslant n/4$ which, together with the fact that $n + 3\delta - 2\sqrt{\delta(n+3\delta)} \leqslant n + 2\delta - 2\sqrt{n\delta}$ if and only if $\delta \geqslant 16n/121$, proves the theorem. $\square$

Theorem 4 is sharp for $n/6 \leqslant \delta \leqslant n/4$, as the following graphs are extremal. For $\delta \leqslant n/4$, let $\tilde{H}(\delta)$ be the family of graphs such that $V = \bigcup_{j=1}^{2}(A_j \cup B_j)$, where $|A_j| = \delta$, $|B_j| = n/2 - \delta$ and $xy \in E$ if and only if $x \in A_1$, $y \in A_2$ or $x \in B_1$, $y \in B_2$ or $x \in A_2$, $y \in B_2$. Then $I = A_1 \cup B_1$ or $A_j \cup B_k$, $j \neq k$, with $i = n/2$. Note that $\tilde{H}(n/4) \cong F(n/4, 2)$.

In the trivial case $\delta = 0$ the graph nK_1 has $i = n$. Otherwise, we are certain that the bounds for $\delta < n/6$ can be improved, especially since the graphs $F(\delta, \ell)$ contain triangles for $\delta < n/4$. In fact we have been unable to find any other examples of triangle-free graphs with $i > n/2$.

We now focus upon the range $n/4 < \delta < 2n/5$. Observe that in this case Proposition 1 implies $i < n - i$. Therefore, in contrast to our previous proofs, it is possible that some vertices of $V - I$ are adjacent to all of I. Again the proof of our main result (Theorem 7) requires two preliminary lemmas.

Lemma 5. *Let G be a simple, triangle-free graph of order n and minimum degree $\delta > n/4$. If a pair of non-adjacent vertices in $V - I$ have neighbourhoods in I with union of order $w < i$, then $w \leqslant n - i - \delta$.*

Proof. For any pair of non-adjacent vertices $x_1, x_2 \in V - I$ construct the sets W_j and X_j for $j = 1, 2$, as above. If $w = |W_1 \cup W_2| < i$ then $I - (W_1 \cup W_2) \neq \emptyset$. Therefore we can form a maximal independent set I^* of G in exactly the same way as in the proof of Lemma 3, yielding the conclusion that $w \leqslant n - i - \delta$. □

Lemma 6. *Let G be a simple, triangle-free graph of order n and minimum degree $\delta > n/4$. If no vertex of $V - I$ is adjacent to all of I then $i \leqslant 3n/4 - \delta$.*

Proof. Choose $x \in V - I$ such that $w = |N(x) \cap I|$ is maximal, and form the sets W, X and R as previously. Then $R \cup (I - W)$ is maximal independent for G, so $|R| + (i - w) \geqslant i$ and thus

$$|X| \geqslant |R| \geqslant w. \tag{3}$$

We have $I - W \neq \emptyset$ by the conditions of the lemma. Also $N(I - W) \subseteq V - I - X$, with each vertex of $I - W$ having at least δ neighbours, so

$$|V - I - X| \geqslant \delta. \tag{4}$$

Now let $Z = N(x) \cap (V - I)$. For $Z = \emptyset$ we have $w \geqslant \delta$, and using (3) and (4) we obtain

$$n - i = |X| + |V - I - X| \geqslant \delta + \delta,$$

which gives $i \leqslant n - 2\delta = (n - \delta) - \delta < 3n/4 - \delta$.

For $Z \neq \emptyset$, if $w \leqslant i/2$, then since the average degree of a vertex of $V - I$ in I is at least $i\delta/(n - i)$, we must have $i\delta/(n - i) \leqslant w \leqslant i/2$. This again implies $i \leqslant n - 2\delta < 3n/4 - \delta$.

If $w > i/2$, then as G is triangle-free, no vertex of Z can have a common neighbour in I with x, so each vertex of Z has at most $i - w$ neighbours in I. In addition, all vertices of $V - I - Z$ have at most w neighbours in I. Therefore an upper bound for $e(V - I, I)$, the number of edges between $V - I$ and I, is given by

$$|Z|(i - w) + (n - i - |Z|)w = |Z|(i - 2w) + w(n - i).$$

Given that $w > i/2$, the right-hand side of this expression is a decreasing function of $|Z|$. Thus we may assume that $|Z|$ is minimum possible, i.e. $|Z| = \delta - w$, so

$$e(V - I, I) \leqslant (\delta - w)(i - 2w) + w(n - i) = 2w^2 + w(n - 2i - 2\delta) + i\delta.$$

Each vertex of I has at least δ neighbours in $V - I$, so we must have $2w^2 + w(n - 2i - 2\delta) + i\delta \geqslant i\delta$, which in turn implies $w \geqslant i + \delta - n/2$. Applying (3) we obtain

$$|X| \geqslant w \geqslant i + \delta - n/2. \tag{5}$$

Finally, combining (4) and (5) we get

$$n - i = |X| + |V - I - X| \geqslant (i + \delta - n/2) + \delta,$$

which gives $i \leqslant 3n/4 - \delta$. □

Theorem 7. *Any simple, triangle-free graph G of order n and minimum degree δ satisfies*

$$i \leqslant \begin{cases} 3n/4 - \delta & \text{if } n/4 < \delta \leqslant n/3, \\ (2n-\delta)/4 & \text{if } n/3 \leqslant \delta < 2n/5. \end{cases}$$

Proof. Suppose $n/4 < \delta < 2n/5$ and let $S = \{v \in V - I : N(v) \supseteq I\}$. If $i \leqslant \max\{3n/4 - \delta, \delta\}$ then the conclusion of the theorem holds, so henceforth we may assume that $i > \max\{3n/4 - \delta, \delta\}$. Therefore $S \neq \emptyset$ by Lemma 6. As G is triangle-free then each $v \in S$ must satisfy $N(v) \cap (V - I) = \emptyset$, so S is a set of isolated vertices in $G[V - I]$.

If $S = V - I$ then since $n - i > i > \delta$, the graph G must be complete bipartite with all vertices of degree greater than δ, an obvious contradiction. We conclude that $V - I - S \neq \emptyset$.

For all $x_j \in V - I - S$, form the sets W_j, X_j and R_j as previously. The set $R_j \cup (I - W_j)$ is maximal independent for G, so $|R_j| + (i - |W_j|) \geqslant i$, implying

$$|X_j| \geqslant |R_j| \geqslant |W_j|. \tag{6}$$

Suppose that some $x_j \in V - I - S$ has no neighbours in $V - I - S$, so $|W_j| \geqslant \delta$. Now $I - W_j \neq \emptyset$ since $x_j \in V - I - S$, and $N(I - W_j) \subseteq V - I - X_j$ with each vertex of $I - W_j$ having at least δ neighbours. Applying (6) we get

$$n - i = |X_j| + |V - I - X_j| \geqslant \delta + \delta.$$

Rearranging this inequality gives $i \leqslant n - 2\delta$, but $n - 2\delta < 3n/4 - \delta$ for $\delta > n/4$, which contradicts our initial assumption. Thus $|W_j| < \delta$ for all j. Consequently all vertices of $V - I - S$ have at least one neighbour in $V - I - S$; we label this condition as (†).

If some $x_j \in V - I - S$ satisfies $|N(x_j) \cap (V - I - S)| > n - 2i$, then we can form an independent set I^* of G, containing x_j and any member of S, with $|I^*| < n - (n - 2i) - i = i$, thereby contradicting the minimality of I. Thus all $x_j \in V - I - S$ must satisfy $|N(x_j) \cap (V - I - S)| \leqslant n - 2i$, and using (6) we deduce that

$$|X_j| \geqslant |W_j| \geqslant \delta - n + 2i. \tag{7}$$

Now suppose that a pair of adjacent vertices $x_1, x_2 \in V - I - S$ have (necessarily disjoint) neighbourhoods in I such that $|W_1| + |W_2| < i$. Then $I - (W_1 \cup W_2) \neq \emptyset$ and $N(I - (W_1 \cup W_2)) \subseteq V - I - (X_1 \cup X_2)$, with each vertex of $I - (W_1 \cup W_2)$ having at least δ neighbours. As $W_1 \cap W_2 = \emptyset$ then $X_1 \cap X_2 = \emptyset$, so $|X_1 \cup X_2| = |X_1| + |X_2|$. Combining these observations with (7), we obtain

$$n - i = |X_1| + |X_2| + |V - I - (X_1 \cup X_2)| \geqslant 2(\delta - n + 2i) + \delta,$$

which yields the upper bound $i \leqslant 3(n - \delta)/5$. However, $3(n - \delta)/5 \leqslant 3n/4 - \delta$ for $\delta \leqslant 3n/8$ and $3(n - \delta)/5 \leqslant \delta$ for $\delta \geqslant 3n/8$, which contradicts our initial assumption. We conclude that any pair of adjacent vertices of $V - I - S$ must have disjoint neighbourhoods in I with orders summing to exactly i, and label this condition as (‡).

Note that X_j is an independent set for all j because if not, any pair of adjacent vertices therein would fail condition (‡). Hence by (†) all vertices of X_j have at least one neighbour in $V - I - S - X_j$. We are left to consider two possibilities.

First suppose that all vertices of $V - I - S - X_j$ have at least one neighbour in X_j. As $N(X_j) \cap I = W_j$, then in order for (‡) to hold, each vertex of $V - I - S - X_j$ must be adjacent to all of $I - W_j$. Clearly all vertices of S are adjacent to all of $I - W_j$, so we see that the subgraph induced by the vertex sets $V - I - X_j$ and $I - W_j$ is complete bipartite. Therefore $V - I - X_j$ must be an independent set, because the addition of any edge to a complete bipartite subgraph of G creates a K_3.

Now $N(S) = I$ and $N(V - I - S - X_j) \cap (V - I) = X_j$, so $V - I - X_j$ is maximal independent for G, implying $n - i - |X_j| \geqslant i$, which is $|X_j| \leqslant n - 2i$. Using this inequality in conjunction with (7) we have

$$n - 2i \geqslant |X_j| \geqslant \delta - n + 2i,$$

which gives $i \leqslant (2n - \delta)/4$.

If not, there must exist some $x_\ell \in V - I - S - X_j$ with no neighbours in X_j for some j. In this case the condition (†), together with the fact that S is a set of isolated vertices in $G[V - I]$, implies the existence of some $x_m \in V - I - S - X_j$ with $x_\ell x_m \in E$. Note that x_m has no neighbours in X_j either, for otherwise by (‡) it must be adjacent to all of $I - W_j$

and so have at least one common neighbour with x_ℓ, a contradiction. By (‡), $W_\ell \cap W_m = \emptyset$ and $W_\ell \cup W_m = I$. Thus in particular $[W_\ell \cap (I - W_j)] \cup [W_m \cap (I - W_j)] = I - W_j$ with

$$|W_\ell \cap (I - W_j)| + |W_m \cap (I - W_j)| = i - |W_j|. \tag{8}$$

Since neither x_ℓ nor x_m is adjacent to all of $I - W_j$, both pairs of non-adjacent vertices x_ℓ, x_j and x_m, x_j have neighbourhoods in I with union of order less than i. Applying Lemma 5, these neighbourhood unions must have order at most $n - i - \delta$, so

$$|W_\ell \cap (I - W_j)| \leqslant n - i - \delta - |W_j| \quad \text{and} \quad |W_m \cap (I - W_j)| \leqslant n - i - \delta - |W_j|. \tag{9}$$

Combining (8) and (9) we obtain $i - |W_j| \leqslant 2(n - i - \delta - |W_j|)$, which gives

$$\begin{aligned} i &\leqslant 2(n-\delta)/3 - |W_j|/3 \\ &\leqslant 2(n-\delta)/3 - (\delta - n + 2i)/3 \quad \text{by (7)} \\ &= n - \delta - 2i/3. \end{aligned}$$

By rearranging we recover the upper bound $i \leqslant 3(n-\delta)/5$, but $3(n-\delta)/5 \leqslant 3n/4 - \delta$, δ, a contradiction.

Recalling our initial conditions, the proof of the theorem is completed by observing that $(2n-\delta)/4 \leqslant 3n/4 - \delta$ for $\delta \leqslant n/3$ and $(2n-\delta)/4 > \delta$ for $\delta < 2n/5$. □

In spite of Theorem 7 being considerably stronger than its counterpart for general graphs, we do not believe it to be best possible. Our work with triangle-free graphs over this range of δ suggests that the following lower bound may be significant.

Theorem 8. *For each rational number $p/q \in [\frac{1}{4}, \frac{1}{3}]$ there is a triangle-free graph G of order n with minimum degree $\delta = pn/q$ and $i = n - 2\delta$.*

Proof. For positive integers a, b with $a \leqslant b$, let $\hat{H}(a,b)$ be the family of graphs such that $V = \bigcup_{j=1}^{4}(A_j \cup B_j)$, where $|A_j| = a$, $|B_j| = b$ and $xy \in E$ if and only if $x \in A_j$, $y \in B_j$ or $x \in A_j$, $y \in B_{j \,(\mathrm{mod}\, 4)+2}$ or $x \in B_j$, $y \in B_{j \,(\mathrm{mod}\, 4)+1}$. Then the graphs $\hat{H}(a,b)$ have order $n = 4(a+b)$, minimum degree $\delta = 2b$, and satisfy $i = 4a = n - 2\delta$. Take $a = (q-2p)n/4q$ and $b = pn/2q$. □

In the light of Theorem 8, we conjecture that $i \leqslant n - 2\delta$ for $n/4 < \delta \leqslant n/3$ with $\hat{H}(a,b)$ extremal, and that $i \leqslant \delta$ for $n/3 \leqslant \delta < 2n/5$ with $K_{\delta,n-\delta}$ extremal.

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